University of Messina - Ph.D. Program in Mathematics

Separation Theorems and Applications

(Prof. Filippo D. Cammaroto)

(30 hours)

1. The Radial Kernel of a Set

Radial sets at a point. Minkowski functional associated with a set that is radial at the origin and its properties. Radial kernel of a set. Properties of the radial kernel. Radial kernel of a convex set. Extremal values of real linear functionals on the radial kernel of a set.

2. The Hahn–Banach Theorem and Separation Theorems

The Hahn–Banach Theorem in real vector spaces. The Hahn–Banach Theorem in real and complex vector spaces and its first corollaries. Separating hyperplanes. Pairs of separated sets in a vector space. Pairs of strictly separated sets in a vector space. Separation theorem in vector spaces. The Hahn–Banach Theorem in locally convex topological vector spaces and its consequences. Separation theorem for two nonempty convex subsets of a topological vector space, at least one of which has nonempty interior. Strict separation theorem for two nonempty convex subsets of a topological vector space and its corollaries. Strict separation theorem for absolutely convex sets.

3. Relative Interior and Pseudo-relative Interior in Banach Spaces

Relative interior of a set in a vector space. Separation theorem in $\mathbb{R}^n$ for two convex sets with disjoint relative interiors. Fenchel duality and the sum rule in $\mathbb{R}^n$. Computation of the relative interior of a convex subset of $\mathbb{R}^n$. Pseudo-relative interior of a set. Example of a set with empty pseudo-relative interior. Inclusion relations among the different notions of interior of a set. Computation of the relative interior and the pseudo-relative interior in Banach spaces. Examples.

4. Quasi-Relative Interior and Applications

Convex cones in vector spaces. Partial order induced by a convex cone. Quasi-relative interior of a convex subset of a topological vector space. Properties of the quasi-relative interior of a set. Relationships among the different notions of interior of a set. Example of a set with empty quasi-relative interior. Application of separation theorems to a strong duality theorem. Separation theorem for convex sets involving their quasi-relative interior. Research perspectives and possible developments.

References

- **H. Brezis**, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer.
- **J.M. Borwein, R. Goebel,** *Notions of Relative Interior in Banach Spaces*, Journal of Mathematical Sciences **115** (2003), 2542–2553.
- **J.M. Borwein, A.S. Lewis,** *Partially Finite Convex Programming, Part I: Quasi-Relative Interiors and Duality Theory*, Mathematical Programming **57** (1992), 15–48.
- **J. Jahn,** *Introduction to the Theory of Nonlinear Optimization*, Springer, Berlin, 1994.
- **F. Cammaroto, B. Di Bella,** *Separation Theorem Based on the Quasi-Relative Interior and Application to Duality Theory*, Journal of Optimization Theory and Applications **125** (1), 223–229, 2005.

- **F. Cammaroto, B. Di Bella,** *On a Separation Theorem Involving the Quasi-Relative Interior*, Proceedings of the Edinburgh Mathematical Society **50**, 605–610, 2007.