



University of Messina - Ph. D. Program in Mathematics and Computer Science
Variational Methods for Partial Differential Equations
(Prof. Filippo D. Cammaroto)
(30 ore)

1. Gâteaux and Fréchet Differentiable Functionals on Banach Spaces

Introduction to variational methods. Gâteaux differentiable functionals defined on Banach spaces. Gâteaux derivative and its fundamental properties. Fréchet differentiable functionals defined on Banach spaces. Fréchet derivative and its fundamental properties. Comparison between the two notions of differentiability. Chain rule for differentiable functionals. Mean Value Theorem for Fréchet differentiable functionals. Product rule for differentiable functionals. Second-order Gâteaux and Fréchet derivatives. Necessary conditions for minima and existence results.

2. Sobolev Spaces

The Sobolev space $W^{1,p}([a,b])$. Completeness, reflexivity and separability of Sobolev spaces. Hölder spaces. The space $W_0^{1,p}([a,b])$. Poincaré inequality. Sobolev embedding theorems. The Sobolev spaces $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$: completeness, reflexivity, separability and embedding theorems. Differentiability of the norm. Variable exponent Sobolev spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$. The modular of the space and its properties.

3. Variational Methods: A First Example

Study of a second-order ordinary boundary value problem. Functionals associated with the problem. Minimization of coercive functionals. Existence theorem for the boundary value problem.

4. A General Variational Principle and Applications

Parameter-dependent minimization problems. Functionals of the form $\Phi + \lambda\Psi$. Ricceri's general variational principle. Conditions ensuring the existence of critical points and multiplicity results. Some applications to nonlinear elliptic problems. Existence of infinitely many solutions for a two-point boundary value problem with homogeneous Dirichlet boundary conditions. Existence of infinitely many solutions for the Dirichlet problem involving the p -Laplacian.

5. Variational Methods for a Kirchhoff-Type Problem

Ricceri's three critical points theorem for functionals of a suitable class. Properties of the $p(x)$ -Laplacian operator. Two existence results providing three solutions for a Kirchhoff-type problem involving the $p(x)$ -Laplacian.

6. Variational Methods for a Problem Involving the Fractional Laplacian

Fractional Sobolev spaces $W^{s,p}$. Gagliardo's definition and the main properties. Embedding theorems. The case $p=2$ and its connection with the fractional Laplacian. Definition of $W^{s,2}$ by means of the Fourier transform and its equivalence with Gagliardo's definition. The fractional Laplacian operator. Equivalence among different definitions of the fractional Laplacian. Study of a problem involving the fractional Laplacian by variational methods: direct minimization approach and the Mountain Pass Theorem.

References

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- Z. Denkowski, S. Migórski, N. S. Papageorgiou, *An Introduction to Nonlinear Analysis*, Springer, New York, 2003.
- B. Ricceri, *A General Variational Principle and Some of Its Applications*, *Journal of Computational and Applied Mathematics* **113** (2000), 401–410.
- B. Ricceri, *A Further Three Critical Points Theorem*, *Nonlinear Analysis* **71** (9), 4151–4157, 2009.
- F. Cammaroto, A. Chinnì, *Infinitely Many Solutions for a Two-Point Boundary Value Problem*, *Far East Journal of Mathematical Sciences* **11** (1), 41–51, 2003.
- F. Cammaroto, A. Chinnì, B. Di Bella, *Infinitely Many Solutions for the Dirichlet Problem Involving the p -Laplacian*, *Nonlinear Analysis* **61**, 41–49, 2005.
- F. Cammaroto, L. Vilasi, *Multiple Solutions for a Kirchhoff-Type Problem Involving the $p(x)$ -Laplacian Operator*, *Nonlinear Analysis* **74**, 1841–1852, 2011.
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