University of Messina - Master's Degree in Mathematics

Foundations of Advanced Analysis - Mod. A

(Prof. Filippo D. Cammaroto)

Academic Year 2025/2026

1. Integration on a Measure Space (9 hours)

Simple functions. Approximation of non-negative measurable real-valued functions by simple functions. Integral of simple functions. Properties of the integral of simple functions. Properties holding almost everywhere. Integral of everywhere-defined or almost everywhere-defined non-negative measurable functions. Properties of the integral. Beppo Levi's Monotone Convergence Theorem. Integration of series. Distributive property of the integral with respect to the measure. Measures on $\wp(M)$, where M is a countable set. μ-quasi-integrable functions and their integral. μ-integrable functions. Conditions for μ-integrability. Properties of the integral. Examples of integration. Integral over a subset of Ω. Signed measures with density with respect to a measure μ. Properties of the signed measure $f\mu$. Transformation of integrals from one measure space to another.

2. Integration with Respect to a Product Measure (8 hours)

Product of σ-algebras. Generators of the product of σ-algebra. Product of σ-finite measures. Tonelli's Theorem. Fubini's Theorem. Tonelli's and Fubini's Theorems for the Lebesgue measure m_{p+q}.

3. The Spaces $L^p(\mu)$ – Part I (4 hours)

μ-measurable p-integrable functions. Hölder's and Minkowski's inequalities. Clarkson's inequalities. The spaces $L^p(\mu)$ $(0<p<+\infty)$. The space $L^\infty(\mu)$. Completeness of $L^\infty(\mu)$.

4. Various Modes of Convergence of Sequences of Measurable Real-Valued Functions (8 hours)

Fatou's Lemma and some of its consequences. Almost everywhere convergence. Convergence in the mean of order p. Dominated Convergence Theorem. Completeness of $L^p(\mu)$ $(0<p<+\infty)$. Relations between convergence in the mean and almost everywhere convergence. Quasi-uniform convergence. Characterization of quasi-uniform convergence. Severini–Egorov Theorem. Convergence in measure. Weyl–Riesz criterion for convergence in measure. Summary of the relationships among the various notions of convergence.

5. The Spaces $L^p(\mu)$ – Part II (9 hours)

Inclusions among L^p spaces. Characterization of the inclusion $L^p(\mu)\subseteq L^q(\mu)$. Topological dual of L^p spaces for $1\le p\le+\infty$. Reflexivity of L^p spaces for $1\le p\le+\infty$. Density theorems in L^p spaces. Separability of L^p spaces for $1\le p\le+\infty$.

6. Measures with Density (6 hours)

Signed measures with density with respect to a measure μ. Integration with respect to a measure with density. Signed measures absolutely continuous with respect to a measure μ. Absolute continuity in the sense of Vitali and in the sense of Caccioppoli. Comparison between the different notions of absolute continuity. Radon–Nikodym Theorem. Uniqueness of the density.

7. Characterization of Convergence in the Mean: Vitali's Theorem (4 hours)

Equi-absolute continuity in the sense of Vitali and in the sense of Caccioppoli. Vitali's Theorem, together with its variants and consequences. Characterization of convergence in the mean through the convergence of norms.