University of Messina – Bachelor's Degree in Mathematics

Mathematical Analysis III

(Prof. Filippo D. Cammaroto)

Academic Year 2025/2026

1. Sequences of Sets (2 hours)

Upper and lower limits of a sequence of sets. Convergent sequences of sets. Monotone sequences of sets. Distance between two sets in a metric space. Sets at zero distance. Continuity of the distance function from a fixed set. Sequences of sets in a topological space exhausting their limit. Intervals and multi-intervals in $\mathbb{R}^h$. Every open subset of $\mathbb{R}^h$ can be exhausted by a sequence of multi-intervals.

2. Lebesgue Measure (6 hours)

Properties of the family of multi-intervals in $\mathbb{R}^h$. Elementary measure of multi-intervals and its properties. Lebesgue measure of bounded open subsets of $\mathbb{R}^h$. Properties of the Lebesgue measure of bounded open sets. Lebesgue measure of bounded closed subsets of $\mathbb{R}^h$. Properties of the Lebesgue measure of bounded closed sets. Lebesgue measurable bounded subsets of $\mathbb{R}^h$. Properties of the family of bounded Lebesgue measurable sets. Operations and order in the extended real line. Lebesgue measurable subsets of $\mathbb{R}^h$. Properties of the family of Lebesgue measurable sets. Properties of the Lebesgue measure on measurable sets.

3. Abstract Measure Theory (8 hours)

σ-algebras and their properties. Trace of a σ-algebra. Generators of a σ-algebra. Dynkin systems and their generators. Measures and their properties. Uniqueness theorem for measures. σ-finite measures. Measurable spaces. Measure spaces. Complete measure spaces. Completion of a measure space. Signed measures and their properties. Positive, negative and total variation of a signed measure. Hahn Decomposition Theorem. Jordan Decomposition Theorem. Mutually singular measures. Further properties of signed measures.

4. Borel Sets of a Topological Space (3 hours)

Review of topology: topologies, bases, generated topology, neighborhoods and relative topology. Borel sets of a topological space. Borel sets of a subspace. Generators of the Borel σ-algebra $\mathscr{B}_h$ of $\mathbb{R}^h$. Inclusion $\mathscr{B}_h \subseteq \mathscr{L}_h$. The measure space $(\mathbb{R}^h, \mathscr{L}_h, m_h)$ as the completion of $(\mathbb{R}^h, \mathscr{B}_h, \lambda_h)$. Metric and Borel structure of the extended real line. Borel measures.

5. Measurable Functions (6 hours)

Measurable functions. Necessary and sufficient condition for measurability. σ-algebra generated by a family of functions. Measurability of composite functions. Image measure induced by a measurable function. Restrictions and extensions of measurable functions. Translation invariance of the Lebesgue measure. Measurable real-valued functions. Characterizations of measurable real-valued functions. Algebraic operations on measurable functions. Simple functions. Approximation of non-negative measurable functions by simple functions.

6. Non-Measurable Sets (5 hours)

Every Lebesgue measurable subset of $\mathbb{R}^h$ with positive measure contains a non-Lebesgue measurable subset (Vitali's construction). The identically zero measure is the only translation-invariant measure defined on $\wp(\mathbb{R}^h)$ that is finite on bounded sets. The Cantor set. The Cantor–Lebesgue singular function. Existence of Lebesgue measurable sets that are not Borel measurable.

7. Normed Spaces and Elements of Topology (6 hours)

Review of topology: compact and sequentially compact topological spaces, separable spaces, metric spaces, equivalence between compactness and sequential compactness in metric spaces, complete metric spaces, totally bounded metric spaces, characterization of compact metric spaces. Normed spaces. Banach spaces. Linear operators. Linear isomorphisms. Linear functionals. Algebraic dual of a normed space. Continuity of linear operators and functionals. Space of continuous linear operators between normed spaces. Topological dual of a normed space. Operator norm. Riesz Lemma. Riesz characterization of finite-dimensional normed spaces.

8. Weak Topology (6 hours)

Weak topology of a normed space. Canonical embedding and its properties. Weak-star topology of the topological dual of a normed space. Comparison between the strong, weak and weak-star topologies. Comparison among different notions of continuity for mappings between normed spaces. Reflexive Banach spaces. Kakutani's and James' characterizations of reflexive Banach spaces. Further properties of reflexive spaces. Uniformly convex spaces. Milman–Pettis Theorem.

9. Hilbert Spaces (6 hours)

Pre-Hilbert spaces. Cauchy–Schwarz inequality. Characterization of norms induced by inner products. Hilbert spaces. Projection Theorem. Representation of a Hilbert space as the direct sum of a closed vector subspace and its orthogonal complement. Riesz Representation Theorem for continuous linear functionals on Hilbert spaces. Orthonormal sets. Fourier series in Hilbert spaces. Bessel inequality. Parseval identity. Riesz–Fischer Theorem. Existence of orthonormal bases in separable Hilbert spaces.

References

- **H.Brezis**, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer.
- **L.V.Kantorovich, G.P.Akilov**, *Functional Analysis*, Pergamon Press.

- For the part on Measure Theory, additional teaching material is available through the Microsoft Teams course platform.