University of Messina
Ph.D. Program in Mathematics and Computational Sciences
Convex Analysis
(Prof. Filippo D. Cammaroto)
(30 hours)

1. Elements of Convex Analysis and Separation Theorems
Convex sets. Radial sets at a point. Minkowski functional associated with a radial set containing the origin and its properties. Locally convex topological vector spaces. Hahn–Banach theorem in vector spaces and its first corollaries. Separating hyperplanes. Pairs of separated sets in a vector space. Pairs of strictly separated sets in a vector space. Separation theorem in vector spaces. Hahn–Banach theorem in locally convex topological vector spaces and its consequences. Separation theorem in a topological vector space for two nonempty convex sets, at least one of which has nonempty interior. Strict separation theorem in a topological vector space for two nonempty convex sets and its corollaries. Strict separation theorem for absolutely convex sets.

2. Relative, Pseudo-Relative and Quasi-Relative Interiors in Banach Spaces
Relative interior of a set in a vector space. Separation theorem in $\mathbb{R}^n$ for two convex sets with disjoint relative interiors. Fenchel duality and the sum rule in $\mathbb{R}^n$. Computation of the relative interior of a convex subset of $\mathbb{R}^n$. Pseudo-relative interior of a set. Example of a set with empty pseudo-relative interior. Relationships among the different notions of interior. Computation of the relative and pseudo-relative interiors in Banach spaces. Examples. Convex cones in vector spaces. Partial order induced by a convex cone. Quasi-relative interior of a convex subset of a topological vector space. Properties of the quasi-relative interior. Relationships among the various notions of interior. Example of a set with empty quasi-relative interior. Application of separation theorems to a strong duality theorem. Separation theorem involving the quasi-relative interior of convex sets. Further developments and research perspectives.

3. Convex Functions
Convex functions and their fundamental properties. Semicontinuity. Lower semicontinuous convex functions. Continuous convex functions. Conjugate functions. Optimization problems.

4. Calculus in Banach Spaces
Gâteaux differentiable functionals defined on Banach spaces. Gâteaux derivative and its fundamental properties. Fréchet differentiable functionals defined on Banach spaces. Fréchet derivative and its fundamental properties. Comparison between the two notions of differentiability. Chain rule for differentiable functionals. Mean Value Theorem for Fréchet differentiable functionals. Product rule for differentiable functionals. Second-order Gâteaux and Fréchet derivatives. Necessary conditions for minima and existence results. Superposition operator.

5. Calculus of Variations
Functionals arising in the calculus of variations. Fundamental lemmas of the calculus of variations. Euler equation. First integrals. Legendre condition. The isoperimetric problem. Geodesics on a surface.

6. A Complement to Ekeland's Variational Principle and Applications
A complement to Ekeland's variational principle. Applications to Caristi's fixed point theorem and to Gâteaux differentiable functionals. A Liouville-type theorem for harmonic functions on exterior

domains. Constancy criteria for functions defined on Banach spaces and on exterior domains of $\mathbb{R}^n$. Further developments and research perspectives.

References

- **H. Brezis**, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer.
- **J.M. Borwein, R. Goebel**, *Notions of Relative Interior in Banach Spaces,* Journal of Mathematical Sciences, 115, (2003), 2542-2553
- **J.M. Borwein, A.S. Lewis**, *Partially finite convex programming, Part I: Quasi relative interiors and duality theory*, Mathematical Programming, 57, (1992), 15–48
- **J. Jahn,** *Introduction to the Theory of Nonlinear Optimization*, Springer, Berlin, 1994
- **F. Cammaroto, B. Di Bella,** *Separation Theorem Based on the Quasirelative Interior and Application to Duality Theory,* Journal of Optimization Theory and Applications, 125 (1), 223-229, 2005.
- **F. Cammaroto, B. Di Bella,** *On a separation theorem involving the quasi-relative interior,* Proceedings of the Edinburgh Mathematical Society, 50, 605-610, 2007.
- **F. Cammaroto, A. Chinnì,** *A Complement to Ekeland's Variational Principle in Banach Spaces,* Bulletin of the Polish Academy of Sciences. Mathematics, 44 (1), 29-33, 1996.
- **F. Cammaroto, A. Chinnì,** *A Liouville-type Theorem for Harmonic Functions on Exterior Domains,* Journal of Mathematical Analysis and Applications, 244 , 1-9, 2000.